Mean Consensus Time of the Voter Model on Networks Partitioned into Two Cliques of Arbitrary Sizes

Michael T. Gastner and Kota Ishida

Yale-NUS College, Division of Science, 16 College Avenue West, #01-220 Singapore 138527
michael.gastner@yale-nus.edu.sg,
WWW home page: http://michaelgastner.com/

1 Introduction

The voter model is a paradigmatic agent-based model that represents opinion dynamics in social networks. The dynamics of the model consists of repeatedly choosing one agent uniformly at random. The selected agent then copies the current opinion of a randomly selected neighbour. As long as the network is connected and finite, this update rule guarantees that the agents must eventually reach a consensus after a finite time T. The mean consensus time $\langle T \rangle$ depends on the initial distribution of opinions and the network structure.

While early studies of the voter model focused on complete graphs or regular lattices, interest has recently shifted towards networks with more complex topologies, for example networks with a community structure [1], [2], [3]. Here we analyze the voter model on the simplest possible multi-community network: two cliques (i.e. fully connected subgraphs) connected by a small number X of intercommunity edges (Figure 1). Previous work on networks with two equally large cliques has shown that the mean consensus time $\langle T \rangle$ is proportional to the number N of vertices in the network unless the connections between the cliques are extremely sparse [2]. Because $\langle T \rangle \propto N$ is the same scaling relation as in the case of a single-clique network [4], it has been argued that community structure is of limited importance for the voter model. Here we show that, on the contrary, the two-clique topology gives rise to many intriguing features.

2 Results

Let us denote by α the relative fraction of vertices in clique 1. For example, in the network depicted in Figure 1, α is equal to $\frac{7}{12}$. For all values of α, sparsely connected cliques need a long time to reach a consensus, as one might intuitively expect. Counterintuitively, however, additional links between the cliques do not necessarily speed up the consensus (except in the special case $\alpha = \frac{1}{2}$). Instead, numerical simulations (Figure 2) show that there is an optimal intermediate connectivity that minimizes $\langle T \rangle$. The simulations suggest that the optimal number of interclique edges scales as $X_{\min} \propto N^{3/2}$, which puts the optimum between the case of a constant number of interclique edges per agent ($X_{\min} \propto N$) and a complete graph ($X_{\min} \propto N^2$). Hence, to accelerate a consensus between cliques, agents should reach out to members in the other clique, but not to the extent that cliques lose their identity as distinct communities.

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We confirm the numerical results with an equation-based analysis. For the sake of simplicity, we show the equations only for the case of a polarized initial condition (i.e. both cliques are internally unanimous, but there is disagreement between the cliques). Similar results can be derived for other initial conditions. We make two heterogeneous mean-field approximations for the consensus time $\langle T \rangle$:

- a Taylor expansion for small X,

$$\langle T \rangle \approx t_{\text{sparse}} = \frac{\alpha^2(1-\alpha)^2 N}{X d(\alpha,N,X)} \big[2(2\alpha^2-2\alpha+1)X^3 + 2(\alpha^2-\alpha+1)NX^2 + \alpha(1-\alpha)(2\alpha^2-2\alpha+3)N^2X + \alpha^2(1-\alpha)^2N^3\big] \tag{1}$$

  with the auxiliary function

$$\begin{aligned} d(\alpha,N,X) =&(3\alpha^2-3\alpha+1)(2\alpha^2-2\alpha+1)X^2 \\ &+\alpha(1-\alpha)(4\alpha^4-8\alpha^3+11\alpha^2-7\alpha+2)NX \\ &+\alpha^2(1-\alpha)^2(2\alpha^2-2\alpha+1)N^2, \end{aligned} \tag{2}$$

- an adiabatic approximation for large X,

$$\langle T \rangle \approx t_{\text{dense}} = -\frac{\alpha(1-\alpha)N[(2\alpha^2-2\alpha+1)N^2+2X]^2}{\alpha(1-\alpha)N^2[(3\alpha^2-3\alpha+1)N^2+2X]+X^2}[m\ln m + (1-m)\ln(1-m)], \tag{3}$$

  where

$$m = \frac{(\alpha^2N^2-\alpha N+X)}{(2\alpha^2-2\alpha+1)N^2-N+2X}\ . \tag{4}$$

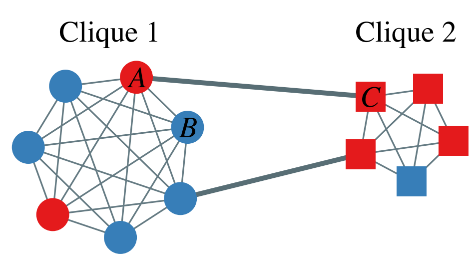

Fig. 1. Small illustrative example of a two-clique network. Each vertex represents an agent that has exactly one of two possible opinions: "red" or "blue". In this example, clique 1 is a complete graph with 7 vertices, whereas clique 2 has only 5 vertices. The cliques are connected by two intercommunity edges (thick lines). In our analysis, we vary the relative sizes of the two communities and the number of intercommunity edges. We apply the update rules of the voter model. That is, we first choose a random focal vertex, for example A in the depicted network. Then we choose a random neighbour of the focal vertex and copy the neighbour's opinion. In our example, if the chosen neighbour is B, A changes its opinion to blue. However, if the chosen neighbour is C, A keeps its current (i.e. red) opinion.

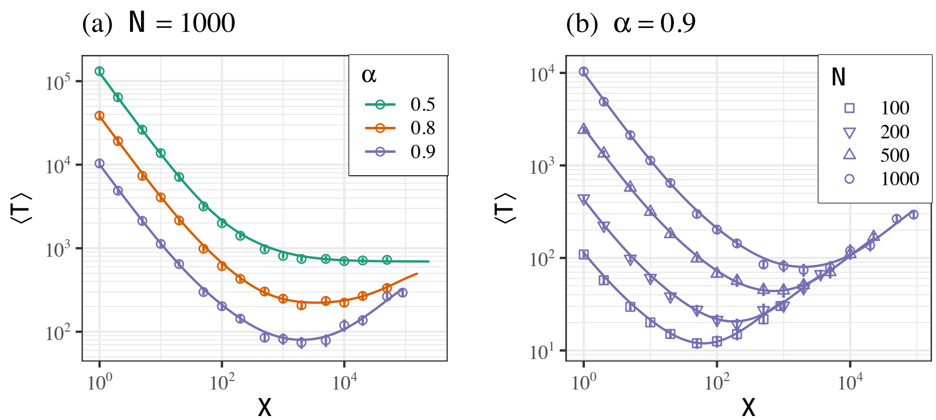

Fig. 2. Mean consensus time $\langle T \rangle$ as a function of the number of interclique edges X. Point symbols represent simulation results. In all simulations, the initial opinions are completely polarized: both cliques are internally unanimous, but there is disagreement between the cliques. The curves are equation-based predictions. In (a), we fix the number of vertices to be $N = 1000$ and vary the fraction α of vertices in the first clique. If $\alpha \neq \frac{1}{2}$, the minimum of $\langle T \rangle$ is attained at an intermediate value of X, where the cliques are neither sparsely nor fully connected. In (b), we fix $\alpha = 0.9$ and vary N. The value $X_{\min}$ that minimizes $\langle T \rangle$ is proportional to $N^{3/2}$.

By interpolating between the two asymptotic approximations, we obtain an equation for $\langle T \rangle$ that is in excellent agreement with the simulations for all values of X,

$$\langle T \rangle = t_{\text{dense}}(X) + t_{\text{sparse}}(X) - \lim_{X' \to \infty} t_{\text{sparse}}(X'). \quad (5)$$

This interpolation is shown by the curves in Figure 2. From equations (1)–(5) it can be shown that $X_{\min} \propto N^{3/2}$ [5], consistent with the numerical results.

Summary. We show that, counterintuitively, the mean consensus time $\langle T \rangle$ is typically not a monotonically decreasing function of interclique connectivity. To minimize $\langle T \rangle$, the optimum number of interclique edges $X_{\min}$ should scale as $X_{\min} \propto N^{3/2}$, where N is the number of vertices. Consequently, to reach a consensus quickly, the agents must strike a balance between a sparse and a dense interclique connectivity.

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