Topology-Preserving Line Densification for Creating Contiguous Cartograms from Density-Equalising Map Projections

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1. Introduction

Cartograms are maps in which the areas of enumeration units (e.g. administrative divisions) are proportional to spatially extensive quantitative data (e.g. population). As the cartogram areas are typically not proportional to land areas, the shapes or the contiguity of the polygons representing the geographic regions must be adjusted. If all adjacent polygons share boundaries even after resizing their areas, the cartogram is called contiguous, such as the example shown in Figure 1. Contiguous cartograms can be constructed using density-equalising map projections. However, invalid topologies, such as intersections or gaps between adjacent polygons, can arise when enumeration units are depicted by polygons using a finite set of vertices connected with straight lines, as required for computer-generated maps. Here we introduce a method for topology-preserving line densification that solves this problem. By carefully augmenting the density of points along boundaries, we ensure that neighbouring regions remain connected but do not overlap, even when discretising their boundaries.

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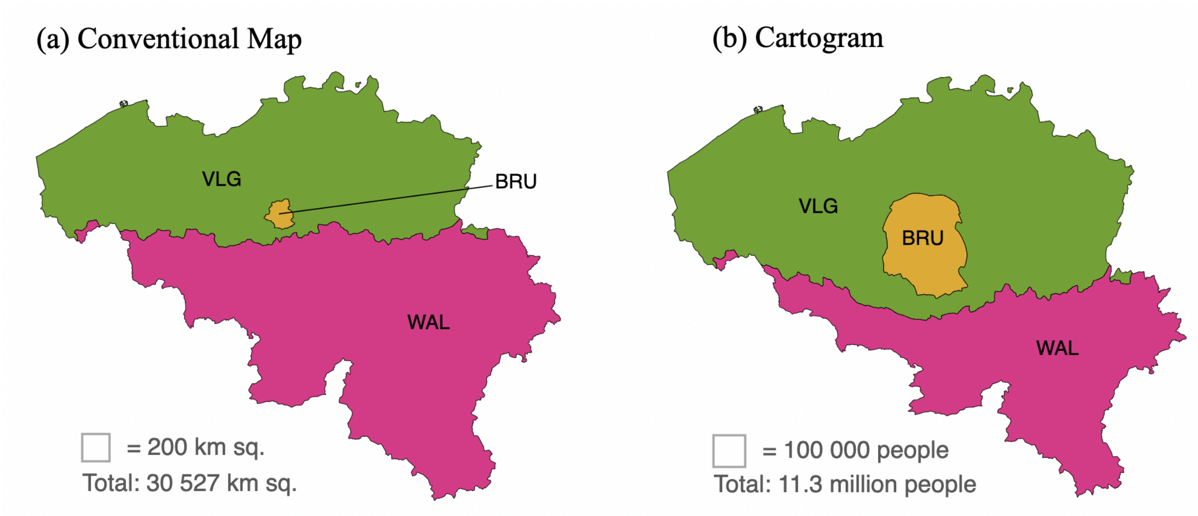

Figure 1: Maps of Belgium. Panel (a) illustrates the country's regions on a conventional map, and panel (b) shows a cartogram representing population. As Brussels, labelled BRU, is the most densely populated region, its area is enlarged in the cartogram.

2. Problem Statement

Density-equalising map projections provide a mathematically principled method for creating contiguous cartograms. In this approach, the input data are expressed as spatial densities (e.g. population per unit area), and each region is resized so that the density becomes uniform across the entire map. Afterwards, the polygon vertices are projected and connected with lines to create the cartogram. A density-equalising map projection Π must satisfy:

$$J_{\Pi}(x, y) = \frac{\rho(x,y)}{\bar{\rho}}, \quad (1)$$

where ρ is the density function, and $\bar{\rho}$ is the average density. The left-hand side of Equation (1) is the Jacobian determinant $J_{\Pi}(x, y) = \frac{\partial \Pi_x}{\partial x}\frac{\partial \Pi_y}{\partial y} - \frac{\partial \Pi_x}{\partial y}\frac{\partial \Pi_y}{\partial x}$ of the projection, which we assume to be differentiable in the entire mapping domain, for example, by applying pycnophylactic interpolation (Gastner et al. 2022). Additionally, we also assume that the density is positive everywhere. This assumption poses no practical restriction because mixtures of positive and non-positive data cannot be proportional to physical areas, and, thus, cartograms would not be suitable for visualisation. Under this premise, Hadamard's global inverse function theorem implies that Π is invertible. Thus, density-equalising map projections theoretically guarantee that no intersections are generated by the transformation.

In practice, however, the boundary of a region needs to be approximated by a polygon with a finite number of points. This restriction can result in intersections emerging after the transformation, even for continuous density-equalising map projections, as illustrated in Figure 2.

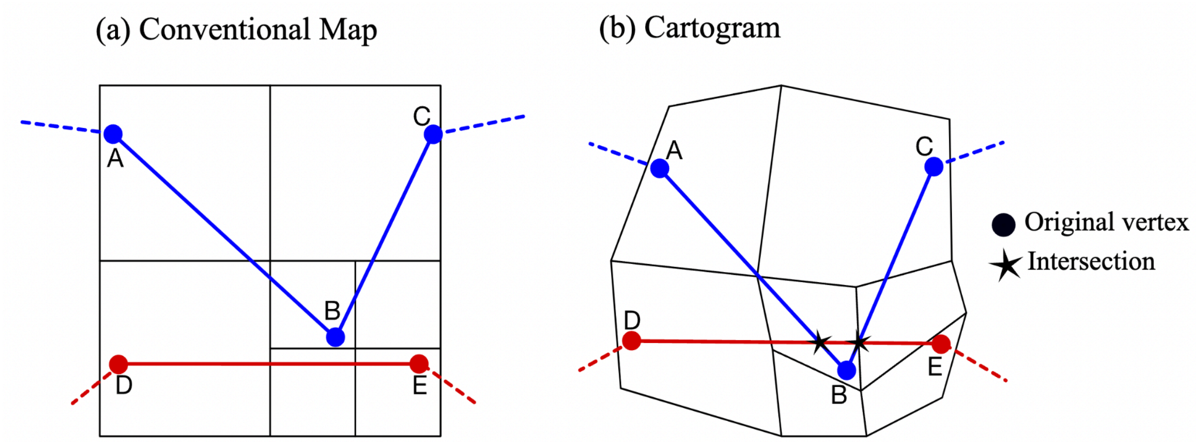

Figure 2: Illustration of a topology violation caused by projecting polylines. The figure depicts two polylines, one coloured blue (ABC) and the other red (DE). Panel (a) illustrates the geometry before a density-equalising map projection, while panel (b) shows the situation afterwards. The projection is indicated by the altered shapes of the black grid lines. The projected positions of the polyline vertices A to E are interpolated on the basis of the quadrilaterals formed by the projected grid lines in (b). Although the blue and red polylines do not intersect before the projection, they cross each other afterwards, resulting in an invalid topology.

3. Related Work

The problem of topology violations arising from cartogram construction is well-documented in the literature. Dougenik et al. (1985) noted that their force-based algorithm can produce intersecting borders, especially around panhandle-shaped regions. Similarly, House and Kocmoud (1998) noted the possibility of intersections during cartogram transformations. As a mitigation strategy, they introduced an intersection-specific penalty term in their relaxation-based method for cartogram construction. Additionally, Tobler (2004) acknowledged the intersection problem and suggested a technique called underrelaxation, which shrinks the displacements caused by the cartogram transformation. However, Tobler noted that, while underrelaxing alleviates the problem, it does not entirely prevent it.

A well-known strategy in data visualisation to address the intersection problem caused by coordinate transformations is line densification, implemented by software tools such as the R package smoothr (Strimas-Mackey 2023). Line densification involves inserting additional vertices along the unprojected polygon edges to increase flexibility of the projected polygons. The conventional approach consists of inserting vertices at regular intervals. However, this method does not optimise the number of insertions. On the one hand, if too few vertices are inserted, the densification might be too sparse to prevent intersections completely. On the other hand, if the densification is too dense, it burdens memory and slows computation.

Dougenik et al. (1985) described a method guaranteeing non-overlapping regions when using their force-based cartogram algorithm. Their method proceeds by dividing the unprojected polygons into convex subpolygons. Yet, as pointed out by Sun (2020), this method's complexity has prevented its implementation in cartogram software. Instead, Sun suggested using quadtrees as auxiliary data structures and inserting additional vertices at intersections between polygons and quadtree edges. However, it is not clear how to interpolate projected coordinates for the polygon vertices that are not on a quadtree edge because, generally, no affine transformation exists that maps the four quadtree cell corners to all four projected counterparts. Therefore, we will now describe an algorithm that supplements the quadtree representation of the density-equalising map projection by using a Delaunay triangulation, guaranteeing topology preservation.

4. Proposed Algorithm

Our algorithm proceeds as follows: We first construct a quadtree from the map, splitting all cells containing 10 or more polygon vertices. This parameter was chosen because it resulted in fast convergence in empirical tests involving 13 real-world data sets of countries and their population by first-level administration. The mean duration for generating the cartogram was 1.15 seconds on a MacBook with an Apple M2 chip, 8 cores and 16 GB memory. The convergence criterion was that the cartogram area of each administrative unit differs by less than 1% from its target value.

Once the quadtree is constructed, we create a Delaunay triangulation of the cell corners (Figure 3a). To ensure that the Delaunay triangles have a maximum aspect ratio of 2:1, we chose the quadtree to be graded, ensuring that the difference in depth between any two neighbouring cells is not more than 1. For every edge in a polygon, we insert points at intersections with any Delaunay triangle boundary. The projected coordinates of the original and added points are then interpolated using the uniquely defined affine transformation that maps the unprojected Delaunay triangles to their respective projections, as indicated in Figure 3(b). Figure 4 shows the application of the line-densification algorithm to real-world data for Belgium.

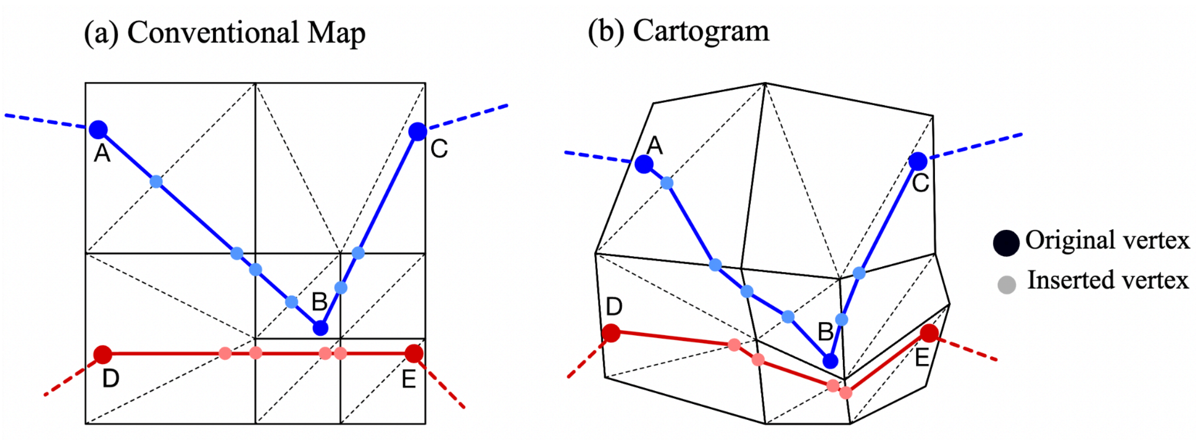

Figure 3: Illustration of our algorithm designed to prevent polyline intersections. (a) The

position of the polyline vertices is identical to Figure 2(a). A quadtree, represented by solid black lines, is created such that the number of points in a quadtree cell does not exceed a specified threshold. For illustration purposes, the threshold is chosen as 1 in this figure. Next, we construct a Delaunay triangulation of the quadtree cell corners, adding the dashed lines to the quadtree grid. Finally, polyline vertices are added at intersections with the Delaunay triangle boundaries. (b) When the vertices added in (a) are reprojected alongside vertices A to E, the red and blue polylines do not intersect. Thus, in contrast to the situation depicted in Figure 2(b), the original topology is maintained.

To establish the guarantee that our proposed algorithm generates non-intersecting projected polygons, we remind the reader that density-equalising map projections are invertible. Consequently, Delaunay triangles cannot overlap after the transformation. Additionally, line segments inside the same triangle will not intersect after the projection because the affine transformation has a non-zero determinant. Furthermore, the affine transformations in adjacent Delaunay triangles join each other continuously, ensuring that no intersections occur on the triangle edges either.

5. Conclusion

In this extended abstract, we addressed the issue of topology violations during the transformation of polylines in the construction of contiguous cartograms using density-equalising map projections. Despite theoretical guarantees, practical implementation often results in intersections due to the approximation of region boundaries with a finite set of points connected with straight lines.

We reviewed existing solutions, noting their limitations, and introduced our topology-preserving line-densification algorithm. By combining quadtree and Delaunay triangulation, our proposed line densification method ensures that polylines transform without intersecting. This advancement contributes to the reliability of contiguous cartogram generators, enhancing their utility in geovisualisation.

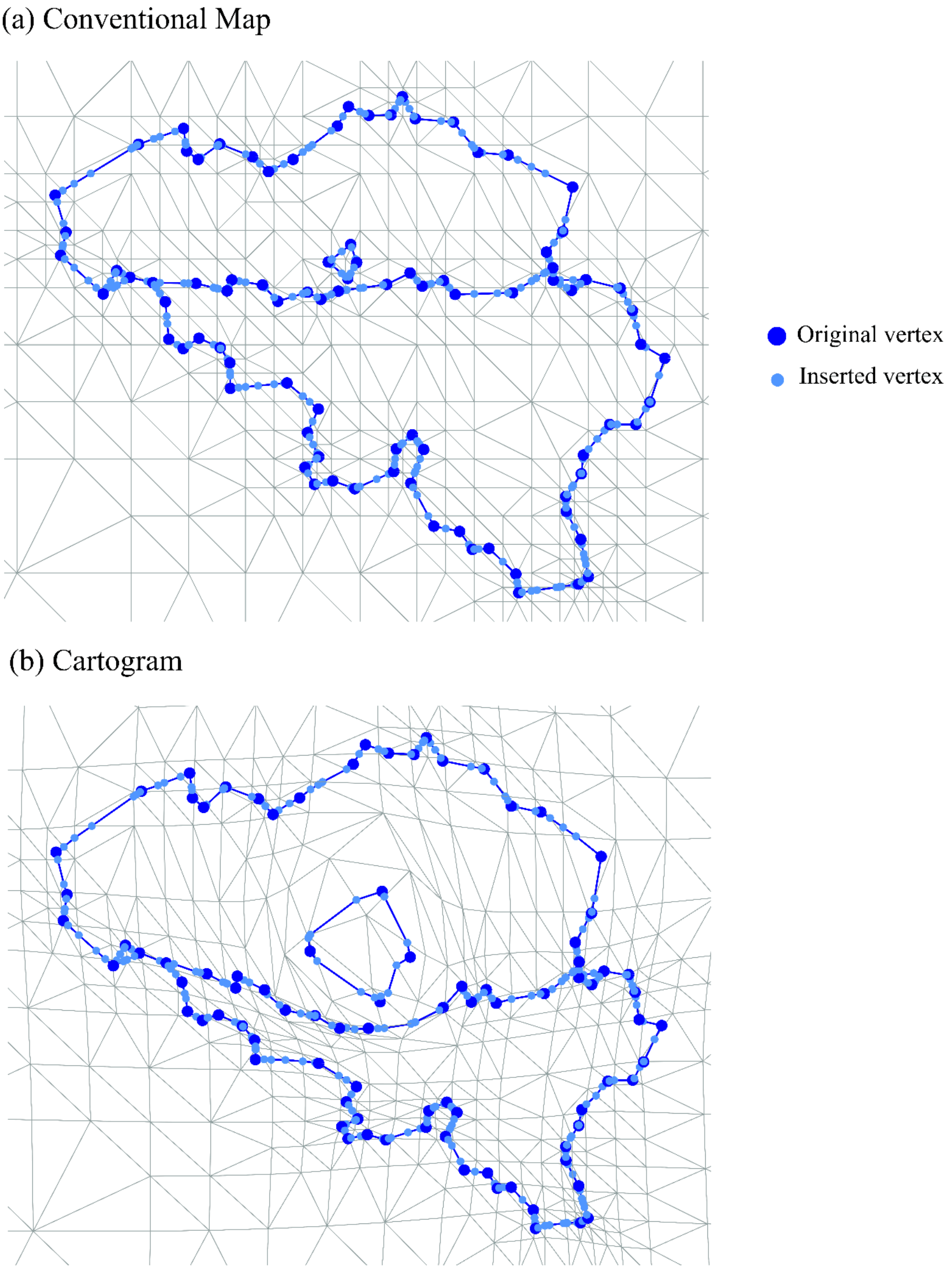

Figure 4: Panel (a) shows the geometry of regions of Belgium on a conventional map before a density-equalising map projection is applied, and panel (b) illustrates a cartogram representing population. The blue lines represent the edges of the polylines, whereas the black lines are the edges of the Delaunay triangles, created from the corners of the quadtree cells. The quadtree itself is created from the vertices of Belgium's conventional map.

6. Acknowledgements

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Biographies

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